



23145318

QP CODE: 23145318

Reg No : .....

Name : .....

**M Sc DEGREE (CSS) EXAMINATION, DECEMBER 2023**

**First Semester**

**CORE - ME010101 - ABSTRACT ALGEBRA**

M Sc MATHEMATICS, M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

563FEA65

Time: 3 Hours

Weightage: 30

**Part A (Short Answer Questions)**

Answer any **eight** questions.

Weight 1 each.

1. Define internal direct product of groups.
2. Define decomposable and indecomposable groups. Give examples.
3. Let  $H$  be a subgroup of a group  $G$ . Prove that  $G$  is an  $H$ -set under conjugation where  $*(h, g) = hgh^{-1}$  for  $g \in G$  and  $h \in H$ .
4. Find the kernel of the homomorphism  $\phi : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_3$ , where  $\phi(1) = 2$ .
5. Prove that every finite p-group is solvable.
6. Find the order and possible number of Sylow 2-subgroups of a group of order 24.
7. Find the sum and product of the polynomials  $f(x) = 4x - 5$  and  $g(x) = 2x^2 - 4x + 2$  in  $\mathbb{Z}_8[x]$
8. Let  $f(x) \in F[x]$  and let  $f(x)$  be of degree 2 or 3. Prove that  $f(x)$  is reducible over  $F$  if and only if it has a zero in  $F$ .
9. Distinguish between a group ring and a group algebra.
10. Find an ideal of  $\mathbb{Z} \times \mathbb{Z}$  which is not a prime ideal.

(8×1=8 weightage)

**Part B (Short Essay/Problems)**

Answer any **six** questions.

Weight 2 each.

11. Let  $X$  be a  $G$ -set. Prove that  $G_x$  is a subgroup of  $G$  for each  $x \in X$ .
12. Derive Burnside's formula





13. Let  $H$  be a  $p$ -subgroup of a finite group  $G$ . If  $p$  divides  $(G : H)$ , then prove that  $N[H] \neq H$ .
14. Let  $G$  be a finite group and let  $|G| = p^n m$ , where  $n \geq 1$  and  $p$  does not divide  $m$ . Show that the Sylow  $p$ -subgroups of  $G$  are precisely those subgroups of order  $p^n$ .
15. Let  $F$  be field of quotients of of an integral domain  $D$  and  $L$  be any field containing  $D$ . Then show that there exists a map  $\psi : F \rightarrow L$  that gives an isomorphism of  $F$  with a subfield of  $L$  such that  $\psi(a) = a$  for  $a \in D$ .
16. State and prove factor theorem.
17. Let  $N$  be an ideal of a ring  $R$ . Then show that  $\gamma : R \rightarrow R/N$  defined by  $\gamma(x) = x + N$  is a ring homomorphism with kernel  $N$ .
18. Show that  $\mathbb{Z}_5[x]/(x^2 + 3x + 2)$  is a field.

(6×2=12 weightage)

### Part C (Essay Type Questions)

Answer any **two** questions.

Weight 5 each.

19. (a) Let  $H$  be a subgroup of a group  $G$ . Prove that the left coset multiplication is well defined by the equation  $(aH)(bH) = (ab)H$  if and only if  $H$  is a normal subgroup of  $G$ .  
(b) Let  $H$  be a normal subgroup of a group  $G$ . Prove that the cosets of  $H$  form a group  $G/H$  under the binary operation  $(aH)(bH) = (ab)H$ .
20. (a) If  $H$  and  $K$  are finite subgroups of a group  $G$ , then prove that  $|HK| = \frac{|H||K|}{|H \cap K|}$ .  
(b) Prove that no group of order 36 is simple.
21. Let  $x$  be a positive integer and let  $a, b \in \mathbb{Z}_m$ . Let  $d = \gcd(a, m)$ . Prove that the equation  $ax \equiv b$  has a solution in  $\mathbb{Z}_m$  if and only if  $d$  divides  $b$ . Also show that when  $d$  divides  $b$ , the equation has exactly  $d$  solutions in  $\mathbb{Z}_m$ .
22. Define ring homomorphism and ring of endomorphisms. Prove that the set  $\text{End}(A)$  of all endomorphisms of an abelian group  $A$  forms a ring.

(2×5=10 weightage)

