**QP CODE: 24044657**

24044657

**Reg No :** .....................

**Name :** .....................

# M.Sc DEGREE (CSS) EXAMINATION, OCTOBER 2024

## Third Semester

M.Sc MATHEMATICS , M.Sc MATHEMATICS (SF)

## CORE - ME010304 - FUNCTIONAL ANALYSIS

2019 ADMISSION ONWARDS

3B7DD300

Time: 3 Hours | Weightage: 30

### Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. State the completion theorem of metric space.
2. If Y and Z are subspaces of a vectorspace X, then show that $Y \cap Z$ is a subspace of X.
3. Prove that the inverse of a linear operator T exists if and only if null space of T is equal to $\{0\}$.
4. Prove that the null space of a linear operator is closed.
5. Let $X$ and $Y$ be finite dimensional vector spaces over the same field and $T: X \to Y$ be a linear operator. Prove that $T$ determines a unique matrix with respect to a basis for $X$.
6. Define an inner product space. Give an Example.
7. Write, Euler formulas for finding the fourier coefficients.
8. State Riesz representation theorem.
9. *Define Hilbert-adjoint operator. Let* $H_1$ *and* $H_2$ *are Hilbert spaces and* $S, T \in B(H_1, H_2)$ *then prove that* $(S+T)^* = S^* + T^*$
10. Define self-adjoint, unitary and normal operators. Prove that a normal operator need not be self-adjoint or unitary.

(8×1=8 weightage)

### Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Show that (i) $x_n \to x, y_n \to y$ implies $x_n + y_n \to x + y$.
    (ii) $\alpha_n \to \alpha$ and $x_n \to x$ implies $\alpha_n x_n \to \alpha x$ .
12. State and prove Riesz's lemma.

13. Define a bounded linear operator on a normed space and prove that $\|T\| = sup\{\|Tx\|/x \in D(T), \|x\| = 1\}$. Also show that this alternate formula for norm satisfies all the conditions of a norm.

14. Let $f : C[a,b] \to R$ be a function defined by $f(x) = \int_a^b x(t)dt$. Is $f$ a bounded linear functional on $C[a,b]$? Justify

15. Let $Y$ be a closed subspace of a Hilbert space $H$. Prove that $Y = Y^{\perp\perp}$.

16. Let $X$ be the inner product space of all real valued continuous functions on $[0, 2\pi]$ with inner product defined by $\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)\, dt$. Show that $u_n(t) = \cos(nt)$ is an orthogonal sequence in $X$.

17. Prove that in every Hilbert space $H \neq \{0\}$, there exists a total orthonormal set.

18. *Let E be an ordered basis of the n-dimensional Euclidean space* $\mathbb{R}^n$ *and* $T$ *be a linear operator on* $\mathbb{R}^n$. *If* $T$ *is represented by the matrix* $T_E$, *then prove that the adjoint operator* $T^\times$ *is represented by the transpose of* $T_E$.

(6×2=12 weightage)

**Part C (Essay Type Questions)**

*Answer any **two** questions.*

*Weight **5** each.*

19. (i) When do you say that two norms are equivalent on a vector space X.
(ii) Prove that on a finite dimensional vector space X, any two norms are equivalent.
(iii) If two norms $||.||, ||.||_0$ on a vector space X are equivalent, show that $||x_n - x|| \to 0$ if and only if $||x_n - x||_0 \to 0$.

20. i)Show that the dual space of $l^1$ is $l^\infty$
ii)Show that dual space $X^l$ of a normed space $X$ is a Banach space.

21. Let $H$ be a Hilbert space.
a) Prove that if $H$ is separable, every orthonormal set in $H$ is countable.
b) Prove that if $H$ contains an orthonormal sequence which is total in $H$, then $H$ is separable.

22. State and prove Hahn-Banach theorem for complex vector spaces.

(2×5=10 weightage)