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Reg. No.....

Name.....

M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY 2016

Third Semester

Faculty of Science

Branch : I (A) Mathematics

MT 03 C12—FUNCTIONAL ANALYSIS

(2012 Admission onwards)

Time : Three Hours

Maximum Weight : 30

Part A

Answer any five questions.  
Each question has weight 1.

1. Give an example of a subspace of  $l^\infty$  which is not closed.
2. Show by an example that a closed and bounded subset of a metric space need not be compact.
3. Let  $X$  be a finite dimensional vector space show that if  $x_0 \in X$  has the property that  $f(x_0) = 0$  for all  $f \in X^*$ , then  $x_0 = 0$ .
4. Show that the space  $l^p$  with  $p \neq 2$  is not a Hilbert space.
5. Define a sesquilinear form on  $X \times Y$ , where  $X$  and  $Y$  are vector spaces.
6. Let  $T: H_1 \rightarrow H_2$  be a bounded linear operator, where  $H_1$  and  $H_2$  are Hilbert spaces, show that  $(T^*)^* = T$ .
7. State Baire's category theorem.
8. Show that  $l'$  is not reflexive.

(5 × 1 = 5)

Part B

Answer any five questions.  
Each question has weight 2.

9. Prove that every finite dimensional normed space is complete.
10. State and prove Riesz's lemma.
11. Define equivalent norms. Show that on a finite dimensional vector space  $X$ , any norm  $\|\cdot\|$  is equivalent to any other norm  $\|\cdot\|_0$ .

Turn over

12. Prove that in an inner product space  $X$ , the norm corresponding to the inner product satisfies  $\|x + y\| \leq \|x\| + \|y\|$ , where the equality sign holds if and only if  $y^2 = 0$  or  $x = cy, c \geq 0$ .
13. State and prove Riesz representation theorem on Hilbert spaces.
14. Prove that the dual space  $X'$  of a normed space  $X$  is separable, then  $X$  itself is separable.
15. Prove that every Hilbert space is reflexive.
16. State Zorn's lemma. Prove that every vector space  $X \neq [0]$  has a Hamel basis.

(5 × 2 = 10)

## Part C

Answer any **three** questions.

Each question has weight 5.

17. Let  $T$  be a linear operator. Then prove that (a) The range  $\mathcal{R}(T)$  is a vector space ; (b) If  $\dim \mathcal{D}(T) = n < \infty$ , then  $\dim \mathcal{R}(T) \leq n$  ; and (c) The null space  $\mathcal{N}(T)$  is a vector space.
18. If  $Y$  is a Banach space, then prove that  $B(X, Y)$  is a Banach space.
19. Show that the dual of  $l^p$  is  $l^q$ , where  $1 < p < \infty$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .
20. (a) Let  $Y$  be any closed subspace of a Hilbert space  $H$ , show that  $H = Y \oplus Z$ , where  $Z = Y^\perp$ .
- (b) Prove that for any subset  $M \neq \phi$  of a Hilbert space  $H$ , the span of  $M$  is dense in  $H$  if and only if  $M^\perp = \{0\}$ .
21. Prove that two Hilbert spaces  $H$  and  $\tilde{H}$ , both real or both complex, are isomorphic if and only if they have the same Hilbert dimension.
22. State and prove the generalized Hahn-Banach theorem.

(3 × 5 = 15)