G 17004058

17004058

Reg. No..............................

Name..................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, DECEMBER 2017

Third Semester

Faculty of Science

Branch I (A)—Mathematics

MT03C12—FUNCTIONAL ANALYSIS

(2012 Admission onwards)

Time : Three Hours | Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Show that range of a linear operator is a vector space.
2. Show that a metric d induced by a norm on a normed space X is translation invariant.
3. Show that in an inner product space $|\langle x, y \rangle| \leq \|x\| \|y\|$.
4. If $x \perp y$ in an inner product space X, show that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.
5. Let H_1 and H_2 be Hilbert spaces, $S : H_1 \rightarrow H_2$ and $T : H_1 \rightarrow H_2$, be bounded linear operators show that $(S+T)^* = S^* + T^*$.
6. Show that if Y is a closed subspace of Hilbert space H, then $Y = Y^{\perp\perp}$.
7. Show that every vector space has a Hamel basis.
8. Show that for every x in a normed space X, $\|x\| = \sup\limits_{\substack{f \in X' \\ f \neq 0}} \dfrac{|f(x)|}{\|f\|}$.

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Let X be a normal space. Prove that there is a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$.

Turn over

10. Prove that in a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

11. State and prove Riesz's lemma.

12. (a) Show that the dual space X' of a normed space X is a Banach space.

    (b) Show that the dual of R^n is R^n.

13. Let $T : H \to H$ be a bounded linear operator on a Hilbert space H. Prove that :

    (a) If T is self-adjoint, $\langle Tx, x \rangle$ is real for all $x \in H$.

    (b) If H is complex and $\langle Tx, x \rangle$ is real for all $x \in H$, the operator T is self-adjoint.

14. Let X be a normed space and let $x_0 \neq 0$ be any element of X. Prove that there exists a bounded linear functional $\tilde{f}$ on X such that $\|\tilde{f}\| = 1, \tilde{f}(x_0) = \|x_0\|$.

15. Prove that every Hilbert space H is reflexive.

16. Define adjoint operator T^x. Show that T^x. is linear and bounded and $\|T^x\| = \|T\|$.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. (a) Prove that if a normed space X has the property that the closed unit ball $M = \{x / \|x\| \leq 1\}$ is compact, then X is finite dimensional.

    (b) Let $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces. Prove that T is continuous if and only if T is bounded.

18. (a) Prove that the dual of l^1 is l^∞.

    (b) Show that the space $c[a, b]$ is not an inner product space.

19. (a) Show that a finite dimensional space is algebraically reflexive.

(b) Find the dual basis of the basis :

$$\{(1,0,0),(0,1,0),(0,0,1)\} \text{ for } R^3.$$

20. (a) State and prove Bessel's inequality.

(b) Give an example of an $x \in l^2$ such that we have strict inequality in part (a).

21. Let H be a Hilbert space. Prove that :

(a) If H is separable, every orthonormal set in H is countable.

(b) If H contains an orthonormal sequence which is total in H, then H is separable.

22. State and prove Hahn-Banach theorem on extension of linear functionals.

(3 × 5 = 15)