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M.Sc. DEGREE (C.S.S.) EXAMINATION, NOVEMBER 2018

Third Semester

Faculty of Science

Branch I (A) : Mathematics

MT 03 C12—FUNCTIONAL ANALYSIS

(2012 Admission onwards)

Time : Three Hours

Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. If $M \neq \phi$ is any subset of a vector space X, show that span M is a subspace of X.

2. Define a bounded linear operator. Also define the norm of such operator.

3. Determine the null space of the operator $T : R^3 \to R^2$ represented by $\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix}$.

4. Show that $C[a, b]$ is not a Hilbert space.

5. Define an orthonormal set. Show that it is linearly independent.

6. If (T_n) is a sequence of bounded linear operators on a Hilbert space and $T_n \to T$, show that $T_n^x \to T^x$.

7. Define a partially ordered set. Also state Zorn's lemma.

8. Define the terms rare, meager and non-meager subsets of a metric space X. Also state Baire's category theorem.

(5 × 1 = 5)

Turn over

Part B

*Answer any **five** questions.*
Each question has weight 2.

9. Prove that every finite dimensional subspace Y of a normed space X is complete.

10. Prove that if a normed space X has the property that the closed unit ball is compact, then X is finite dimensional.

11. If a normed space X is finite dimensional, prove that every linear operator on X is bounded.

12. Show that the dual of l' is l^{∞}.

13. State and prove Bessel inequality.

14. Prove that the adjoint operator T^{X} is linear and bounded and $\left\| T^{X} \right\| = \| T \|$.

15. Show that every Hilbert space is reflexive.

16. Let Y be a proper closed subspace of a normed space X, $x_0 \in X - Y$ be arbitrary and $\delta = \inf_{\tilde{y} \in Y} \| \tilde{y} - x_0 \|$. Show that there exists on $\tilde{f} \in X'$ such that $\left\| \tilde{f} \right\| = 1, \tilde{f}(y) = 0$ for all $y \in Y, \tilde{f}(x_0) = \delta$.

$(5 \times 2 = 10)$

Part C

*Answer any **three** questions.*
Each question has weight 5.

17. (i) State and prove Riesz's lemma.

(ii) Let X and Y be metric spaces and $T : X \to Y$ a continuous map. Show that the image of a compact subset M of X under T is compact.

18. (i) Prove that the vector space $B(X, Y)$ of all bounded linear operators from a normed space X into a normed space Y is itself a normed space with norm defined by $\| T \| = \sup_{\| x \| \neq 0} \frac{\| Tx \|}{\| x \|}$.

(ii) If Y is a Banach space, show that $B(X, Y)$ is a Banach space.

19. (i) State and prove Schwarz and triangle inequalities.

(ii) Show that for a sequence (x_n) in an inner product space. The conditions $\| x_n \| \to \| x \|$ and $< x_n, x > \to < x, x >$ imply convergence $x_n \to x$.

20. Prove that two Hilbert spaces H and $\tilde{\text{H}}$, both real or both complex, are isomorphic if and only if they have the same Hilbert dimension.

21. State and prove Riesz's theorem for functionals of Hilbert spaces.

22. State and prove generalized Hahn-Banach theorem.

(3 × 5 = 15)