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Reg. No............................

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M.Sc. DEGREE (C.S.S.) EXAMINATION, MAY 2018

Fourth Semester

Faculty of Science

Branch—I (A)—Mathematics

MT 04 E 01—ANALYTIC NUMBER THEORY

(2012 Admission onwards)

Time : Three Hours

Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question carries weight 1.

1. Show that $\phi\left(p^{\alpha}\right) = p^{\alpha} - p^{\alpha-1}$ for p, prime and $\alpha \geq 1$.

2. Define Bell series of an arithmetical function. Also Find $\mu_p(x)$.

3. Show that $\sum_{n \leq x} \frac{1}{n} = \log x + c + o\left(\frac{1}{x}\right)$ if $x \geq 1$.

4. Show that $0 \leq \frac{4(x)}{x} - \frac{\vartheta(x)}{(x)} \leq \frac{(\log x)^2}{\sqrt[2]{x}\log^2}$ for $x > 0$.

5. Show that for $x \geq 2, \pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t\log^2 t}\,dt$.

6. Prove that $a \equiv b \pmod{m}$ if and only if a and b give the same remainded when divided by m.

7. Show that if a prime p does not divide a, then $a^{p-1} \equiv 1 \pmod{p}$.

8. Given that $m \geq 1$, $(a, m| = |1$ and $f = \exp_m(a)$. Show that $a^k \equiv a^h \pmod{m}$ if and only if $k \equiv h \pmod{f}$.

(5 × 1 = 5)

Turn over

Part B

Answer any **five** *questions.*
Each question carries weight 2.

9. For $n \geq 1$, prove that $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$.

10. State and prove the generalised inversion formula. Also deduce Generalised Mobius inversion formula.

11. Prove that the set of lattice points visible from the origin has density $6/\pi^2$.

12. Prove that the following relations are logically equivalent :

    (a) $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.$

    (b) $\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$

    (c) $\lim_{r \to \infty} \frac{\psi(x)}{x} = 1.$

13. Prove the following :

    (a) $\hat{a} = \hat{b}$ if and only if $a \equiv b \pmod{m}$.

    (b) Two integers x and y are in the same residue class if and only if $x \equiv y \pmod{m}$.

    (c) The m residue classes $\hat{1}, \hat{2} \ldots. \hat{m}$ are disjoint and their union is the set of all integers.

14. Show that for any prime p, all the co-efficients of the polynomial

    $f(x) = (x-1)(x-2) \ldots. (x-p+1) - x^{p-1} + 1$ are divisible by p.

15. State and prove Chinese remainder theorem.

16. Let p be an odd prime and let d be any positive divisor of $p-1$. Show that in every reduced residue system mod p, there are exactly $\phi(d)$ numbers a such that $\exp(a) = d$.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question carries weight 5.

17. For all $x \geq 1$ show that :

$$\sum_{n \leq x} d(n) = x \log x + (2c-1) + O\left(\sqrt{x}\right),$$ where c is the Euler's constant.

18. (a) Prove that for every $x \geq 1$, $[x]! = \prod_{p \leq x} p^{\alpha(p)}$ where the product is extended over all primes $\leq x$ and $\alpha(p) = \sum_{m=1}^{\infty} \left[\frac{x}{p^m}\right]$.

(b) If $x \geq 2$ $\log [x]! = x \log x - x + O(\log x)$.

(c) For $x \geq 2$ $\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x)$.

where the sum is extended over all primes $\leq x$.

19. State and prove Shapiro's Tauberian theorem.

20. (a) State and prove Wolstenholme's theorem.

(b) Show that the set of lattice points in the plane visible from the origin contains arbitrary large square gaps.

Turn over

21. (a) State and prove Lagrange's theorem for polynomial congruence.

(b) Let f be a polynomial with integer co-efficients, let $m_1, m_2, \ldots m_r$ be positive integers relatively prime in pairs, and let $m = m_1, m_2 \ldots . m_r$. Prove that the congruence $f(x) \equiv 0 \pmod{m}$ has a solution if and only if each of the congruences $f(x) \equiv 0 \pmod{m_i}$ $(i = 1, 2, \ldots r)$ has a solution. Also show that if $V(m)$ and $V(m_i)$ denote the solutions of $f(x) \equiv 0 \pmod{m}$ and $f(x) \equiv 0 \pmod{m_i}$ for $i = 1, 2, \ldots r$, then $V(m) = V(m_1) V(m_2) \ldots V(m_r)$.

22. If $|x| < 1$, prove that $\prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{n=0}^{\infty} p(n) x^n$ where $p(0) = 1$, $p(n)$ denotes the partition function.

(3 × 5 = 15)