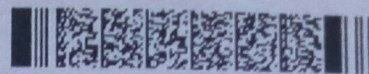


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Reg. No.....

Name.....

M.Sc. DEGREE (C.S.S.) EXAMINATION, APRIL 2019

Fourth Semester

Faculty of Science

Branch I (A)—Mathematics

MT 04 E01—ANALYTIC NUMBER THEORY

(2012 Admission onwards)

Time : Three Hours

Maximum Weight : 30

Part A

Answer any five questions.

Each question has weight 1.

1. Show that Dirichlet multiplication is commutative.
2. Show that Euler totient $\phi(u)$ is multiplicative. Is $\phi(u)$ completely multiplicative. Justify.
3. Show that for $x \geq 1$, $\sum_{n \leq x} \mu(n) \left[\frac{x}{n} \right] = 1$.
4. Define Chebyshev's ψ functions, show that $\psi(x) = \sum_{m \leq \log_2 x} \sum_{P \leq x^{1/m}} \log P$.
5. Show that for $x \geq 2$ $\mathcal{J}(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt$.
6. Show that if $a \equiv b \pmod{m}$ and d/a and d/m , then d/b .
7. Solve the congruence $5x = 3 \pmod{24}$.
8. If g is a primitive root mod P , where p is an odd prime. Show that g^2, g^4, \dots, g^{p-1} are quadratic residues mod P .

(5 × 1 = 5)



Part B

Answer any **five** questions.
Each question has weight 2.

9. If $n \leq 1$ show that $\sum_{d|n} \phi(d) = n$.
10. If f and g are multiplicative functions prove that their Dirichlet product $f * g$ is multiplicative.
11. State and prove Euler's summation formula.
12. For any arithmetical function $\alpha(n)$ Let $A(x) = \sum_{n \leq x} \alpha(n)$, where $A(x) = 0$ if $x < 1$. Assume f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Prove that
- $$\sum_{y < n \leq x} \alpha(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) dt.$$
13. Prove that if $(a, m) = 1$, the linear congruence $ax \equiv b \pmod{m}$ has exactly one solution.
14. State and prove Euler-Fermat theorem.
15. State and prove Wilson's theorem.
16. Let x be an odd integer and $\alpha \geq 3$. Show that $x^{\phi(2^\alpha)/2} \equiv 1 \pmod{2^\alpha}$.

(5 × 2 = 10)

Part C

Answer any **three** questions.
Each question has weight 5.

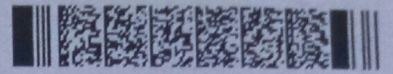
17. If $x \geq 1$, prove the following :

$$(a) \quad \sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right).$$

$$(b) \quad \sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \int (s) + O(x^{-s}) \text{ if } s > 0, s \neq 1.$$

$$(c) \quad \sum_{n > x} \frac{1}{n^s} = O(x^{1-s}) \text{ if } s > 1.$$

$$(d) \quad \sum_{n \leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha) \text{ if } \alpha \geq 0.$$



18. For all $x \geq 1$, show that $\sum_{n \leq x} d(n) = x \log x + (2C - 1)x + O(\sqrt{x})$, where C is Euler's constant.
19. Let P_n denote the n^{th} prime. Show that the following asymptotic relations are logically equivalent :

(i) $\lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = 1.$

(ii) $\lim_{x \rightarrow \infty} \frac{\pi(x) \log \pi(x)}{x} = 1.$

(iii) $\lim_{n \rightarrow \infty} \frac{P_n}{n \log n} = 1.$

20. (a) State and prove Chinese remainder theorem.
(b) Prove that the set of lattice points in the plane visible from the origin contains arbitrary large square gaps.
21. (a) Assume $(a, m) = d$ and suppose $d|b$. Prove that the linear congruence $ax \equiv b \pmod{m}$ has exactly d solutions. Also obtain these solutions.
(b) State and prove Lagrange's theorem for polynomial Congruence.
22. Let P be an odd prime and let d be any positive divisor of $P - 1$. Prove that in every reduced residue system and P there are exactly $\phi(d)$ numbers a such that $\exp_p(a) = d$.

(3 × 5 = 15)

